On the vacuum energy of a spherical plasma shell

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We consider the vacuum energy of the electromagnetic field interacting with a spherical plasma shell together with a model for the classical motion of the shell. We calculate the heat kernel coefficients, especially that for the TM mode, and carry out the renormalization by redefining the parameters of the classical model. It turns out that this is possible and results in a model, which in the limit of the plasma shell becoming an ideal conductor reproduces the vacuum energy found by Boyer in 1968.

I. INTRODUCTION

The present paper is devoted to the discussion of the renormalization of vacuum energy in the presence of boundaries or singular background fields in application to the Casimir effect and it is aimed to partially fill the gap between the two well understood situations. These are, on the one side, the Casimir force between distinct objects which is always finite and, on the other side, the vacuum energy in smooth background fields which can be renormalized by standard methods of quantum field theory. In between these two, the situation is not finally settled. Especially in [12] it was questioned whether boundaries can be incorporated at all into a well posed renormalization program. For instance, it was argued that the process of making the background field concentrated on a surface is not physical.

The aim of the present paper is to discuss an example of a background field concentrated on a surface having both, a well posed renormalization procedure for the vacuum energy and a meaningful physical interpretation. As model we take a spherical plasma shell interacting with the electromagnetic field and we allow for a classical vibrational motion of the shell. The investigation of the plasma shell model was pioneered by Barton [3] and it is aimed to describe the π-electrons in a C_{60}-molecule.

The heat kernel coefficients for such system are known to a large extend. Since the polarizations for the electromagnetic field separate into the usual TE and TM modes, one is faced with two scalar problems, where, however, the s-wave contribution must be dropped. For the TE modes it is a delta function potential on the shell. The corresponding heat kernel coefficients (including the s-wave) were first calculated in [7], later generalized in [10], and the finite part of the vacuum energy was calculated in [16]. For the TM mode the scalar problem corresponds to a δ'-potential on the shell and the corresponding heat kernel coefficients were calculated for a plane shell only, [9]. The problem with the TM mode is that the corresponding spectral problem is not elliptic and that the standard methods do not work. So, for example, for the plane shell even the zeta function cannot be defined [9]. For the spherical shell, the zeta function exists, but, as we will see below, it has double poles. It should be mentioned that the δ'-potential was considered in [15] (where also the relevant literature was collected), however with a coupling different from that following within the plasma shell model (compare the Jost function in (4.41) in [15] with (18) below).

In the present paper we calculate the heat kernel coefficients for the plasma shell model, especially that for the TM modes. Using these, and the simplest possible model for a classical motion of the shell, we construct a consistent scheme for the renormalization. Within this scheme we define the renormalized vacuum energy of the electromagnetic field and calculate it numerically. Also we

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discuss several limiting cases including the limit of the plasma shell becoming an ideal conducting sphere.

Throughout the paper we use units with $\hbar = c = 1$.

II. THE PLASMA SHELL MODEL AND ITS RENORMALIZATION

We consider the plasma shell model investigated, for example, in [3] which is aimed to model the π-electrons in a C_{60}-molecule. These electrons are described by an electrically charged fluid whose motion is confined the shell. Further, the model contains an immobile, overall electrically neutralizing background aimed to describe the carbon atoms and the remaining electrons. The fluid is allowed a non-relativistic motion. Of course, this model is a quite crude simplification, especially because the motion of the electrons should rather follow a relativistic dispersion relation [13, 14]. On the other hand side it appears to be physically meaningful and should therefore result in physically meaningful results for the vacuum energy. For instance, it should allow for a treatment of the vacuum fluctuation of the electromagnetic field coupled to the plasma shell.

The interaction of the plasma shell with the electromagnetic field results in matching conditions on the electromagnetic field across the shell as shown in [3] (and earlier, for a plane sheet, in [2]). These conditions do not depend on the state of the excitations of the fluid. The vacuum energy can be calculated from the fluctuations of the electromagnetic field whereas the fluctuations of the fluid must not be taken into account as shown in [6] (or vice verse). In this setup, the polarizations of the electromagnetic field separate into TE- and TM-modes. For the electric and the magnetic fields the corresponding mode expansions read

$$\begin{aligned} \mathbf{E}^{\mathrm{TE}}(t,\mathbf{r}) &= \sum_{\substack{l\geq 1\\ |m|\leq l}} \int_0^\infty \frac{dk}{\pi}\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t} f_{l,m}(k,r)\mathbf{L}\frac{1}{\sqrt{L^2}}\,Y_{l,m}(\vartheta,\varphi) + c.c.\right), \\ \mathbf{B}^{\mathrm{TE}}(t,\mathbf{r}) &= -\frac{i}{\sqrt{-\Delta}}\,\nabla\times\mathbf{E}^{\mathrm{TE}}(t,\mathbf{r}), \end{aligned} \tag{1}$$

where $\mathbf{L}$ is the orbital momentum operator and $\omega = k$ follows from the wave equation . The radial wave function $f_{l,m}(kr)$ must be regular in the origin and across the shell it must fulfill the matching conditions

$$\begin{aligned} \lim_{r\to R+0} f_{l,m}(kr) - \lim_{r\to R-0} f_{l,m}(kr) &= 0, \\ \lim_{r\to R+0}(rf_{l,m}(k,r))' - \lim_{r\to R-0}(rf_{l,m}(k,r))' &= \Omega R f_{l,m}(kR), \end{aligned} \tag{2}$$

where only the parameter

$$\Omega = \frac{4\pi n e^2}{mc^2} \tag{3}$$

carries information on the properties of the fluid like its density n and mass m. It can be interpreted as a kind of plasma frequency in parallel to the plasma frequency of a dielectric. For C_{60} the corresponding wave lengths is of the order of micrometers. The mode expansions for the TM polarization read by duality

$$\begin{aligned} \mathbf{B}^{\mathrm{TM}}(t,\mathbf{r}) &= \sum_{\substack{l\geq 1\\ |m|\leq l}} \int_0^\infty \frac{dk}{\pi}\frac{1}{\sqrt{2\omega}}\left(e^{-i\omega t} g_{l,m}(k,r)\mathbf{L}\frac{1}{\sqrt{L^2}}\,Y_{l,m}(\vartheta,\varphi) + c.c.\right), \\ \mathbf{E}^{\mathrm{TM}}(t,\mathbf{r}) &= \frac{i}{\sqrt{-\Delta}}\,\nabla\times\mathbf{B}^{\mathrm{TM}}(t,\mathbf{r}). \end{aligned} \tag{4}$$

The matching conditions are different,

$$\begin{aligned} \lim_{r\to R+0}(rg_{l,m}(k,r))' - \lim_{r\to R-0}(rg_{l,m}(k,r))' &= 0, \\ \lim_{r\to R+0} g_{l,m}(kr) - \lim_{r\to R-0} g_{l,m}(kr) &= -\frac{\Omega}{k^2 R}(Rg_{l,m}(k,R))'. \end{aligned} \tag{5}$$

Considered as a scalar problem, the matching conditions (2) of the TE mode are equivalent to a delta function potential $\Omega\delta(r-R)$ in the wave equation and the conditions (5) of the TM mode loosely speaking correspond to the derivative of a delta function. A difference is that in the scalar problems the zeroth orbital momentum, $l=0$, or s-wave contribution is present whereas in the electromagnetic case it is absent, i.e., the sums over l in (1) and in (4) start from $l=1$. In the limit $\Omega\to\infty$ which is formally the ideal conductor limit the boundary conditions (2) and (5) became Dirichlet boundary conditions for TE polarization and Neumann for TM polarization.

We extend this model by allowing for radial vibrations (breathing mode) of the plasma shell. In C_{60} these are determined by the elastic forces acting between the carbon atoms. Without going here in any detail we describe these vibrations phenomenologically by a Hamilton function

$$H_{\rm class}=\frac{p^2}{2m}+\frac{m}{2}\,\omega_b^2\,(R-R_0)^2+E_{\rm rest} \tag{6}$$

with a momentum $p=m\dot{R}$. Here m is the mass of the shell, ω_b is the frequency of the breathing mode, R_0 is the radius at rest and $E_{\rm rest}$ is the energy which is required to bring the pieces of the shell apart, i.e., it is some kind of ionization energy.

Now we consider a system consisting of the classical motion of the shell as described by $H_{\rm class}$ and the vacuum energy $E_{\rm vac}$ of the electromagnetic field interacting with the shell by means of the matching conditions (2) and (5). We assume the classical motion adiabatically slow such that the vacuum energy can be taken as a function of the mountainous radius of the shell, $E_{\rm vac}=E_{\rm vac}(R)$, and we neglect the backreaction of the electromagnetic field on the shell. Under these assumptions the energy of the classical system, $E_{\rm class}(R)=H_{\rm class}$, and the vacuum energy add up to the total energy of the considered system,

$$E_{\rm tot}=E_{\rm class}(R)+E_{\rm vac}(R). \tag{7}$$

Next we consider the ultraviolet divergences of the vacuum energy. These are given in general terms by the heat kernel coefficients a_n (we use the notations of [8] and we can define a 'divergent part' of the vacuum energy which is, as known, not uniquely defined. It depends on the kind of regularization one has to introduce. For instance, in zeta functional regularization, the regularized vacuum energy reads

$$E_{\rm vac}(s)=\frac{\mu^{2s}}{2}\sum_n\omega_n^{1-2s}, \tag{8}$$

where μ is an arbitrary parameter with the dimension of a mass and with a frequency damping function it is,

$$E_{\rm vac}(\delta)=\frac{1}{2}\sum_n\omega_n\,e^{-\delta\omega_n}, \tag{9}$$

where ω_n are the frequencies of the quantum fluctuations of the electromagnetic field. In our problem the spectrum is continuous, but for the moment it is more instructive to keep the notations of a discrete spectrum. In zeta functional regularization, the divergent part reads

$$E_{\rm vac}^{\rm div}(s)=-\frac{a_2}{32\pi^2}\left(\frac{1}{s}+\ln\mu^2\right), \tag{10}$$

where we used the notations of [8] in which the heat kernel expansion reads

$$K(t)\sim\frac{1}{(4\pi t)^{3/2}}\left(a_0+a_{\frac12}\sqrt{t}+a_1\,t+\dots\right). \tag{11}$$

In the scheme with the frequency damping we have

$$E_{\rm vac}^{\rm div}(\delta)=\frac{3a_0}{2\pi^2}\frac{1}{\delta^4}+\frac{a_{1/2}}{4\pi^{3/2}}\frac{1}{\delta^3}+\frac{a_1}{8\pi^2}\frac{1}{\delta^2}+\frac{a_2}{16\pi^2}\ln\delta. \tag{12}$$

The regularizations are removed by $s\to 0$ resp. $\delta\to 0$. These formulas follow, for example, from section 3.4 in [8] for $m=0$.

The idea of the renormalization is to have in the classical energy $E_{\rm class}$ parameters which can be changed in a way to absorb $E_{\rm vac}^{\rm div}$. In the considered model such parameters are the mass m of the shell, the frequency ω_b of the breathing mode, the radius at rest R_0, and the energy $E_{\rm rest}$. Now, whether this is possible, is a matter of the dependence of the heat kernel coefficients, especially of a_2, on the radius R which is the dynamical variable of the classical system. In the considered, very simple model we have only a polynomial dependence on R up to second order in (6). Since we assumed adiabaticity for the motion of the shell we do not have a time dependence in a_2 so that it cannot contain $\dot{R}$. Hence, the kinetic energy remains unchanged and, together with it, the mass m. Only the remaining parameters, ω_b, R_0 and $E_{\rm rest}$ can be used to accommodate the divergent part. In fact, this turns out to be sufficient for the considered model. As it will be seen below, the heat kernel coefficients $a_0, \dots, a_2$ which enter the divergent part, depend on the radius polynomial and at most quadratically. In this way this model is renormalizable.

It should be mentioned that this scheme is equivalent to the corresponding one in quantum field theory with $E_{\rm vac}^{\rm div}$ in place of the counterterms. Also the interpretation of the renormalization is similar. Namely, we argue that the vacuum energy in fact cannot be switched off and what we observe are parameters like, for example in QED, electron mass and charge, after renormalization.

Within this scheme of renormalization, the specific form of the heat kernel coefficients is insignificant. The only what one has to bother of is its dependence on R to fit into the freedom of redefining the parameters in $E_{\rm class}$. If this is the case, one may define a renormalized vacuum energy by means of

$$E_{\rm vac}^{\rm ren} = \lim_{s\to 0} \left(E_{\rm vac}(s) - E_{\rm vac}^{\rm div}(s) \right) \tag{13}$$

(and the same with δ in place of s) and one has now to consider

$$E_{\rm tot} = E_{\rm class} + E_{\rm vac}^{\rm ren} \tag{14}$$

in place of (7). In this way, the question on how to remove the ultraviolet divergences is answered.

It remains, however, the question about the uniqueness of his procedure which comes in from the parameter μ in the zeta functional scheme or from the possibility of a redefinition $\delta \to c\delta$ in the other scheme.

In the case of QED at this place one imposes conditions on the analog of $E_{\rm vac}^{\rm ren}$ in a way, the the mass and the charge take the values one observes experimentally.

In our case a similar scheme is conceivable too. A different scheme, suggested in [7], using the large mass expansion to fix the ambiguity does not work here since the electromagnetic field is massless. A way out could be to look for a minimum of the total energy, $E_{\rm tot}$, (7), which however would imply to take the model (6) seriously. This is not the aim of the present paper. Instead, as a normalization condition we demand that in the limit of the plasma frequency $\Omega \to \infty$, where the matching conditions (2) and (5) turn into that of an ideal conductor, we shall recover the vacuum energy of a conducting spherical shell, i.e., just the quantity which was first calculated by Boyer in [11]. Indeed, as we will see in the next section, this is possible using the freedom of a finite renormalization.

III. THE JOST FUNCTIONS AND THE HEAT KERNEL COEFFICIENTS FOR THE SPHERICAL SHELL

The electromagnetic field interacting with the plasma shell is defined in the whole space and it has a continuous spectrum. In that case the vacuum energy, after the subtraction of the contribution of the empty space, can be represented in the form (see Eq.(3.43) in [8])

$$E_0(s) = -\frac{\cos \pi s}{\pi} \mu^{2s} \sum_{l=1}^{\infty} \nu \int_0^{\infty} dk\, k^{1-2s} \frac{\partial}{\partial k} \ln f_l(ik) \tag{15}$$

with $\nu = l + 1/2$. The arbitrary parameter μ has the dimension of a mass and $f_l(k)$ is the Jost function of the corresponding scattering problem. It is determined by the wave equation and the matching conditions (2) and (5) respectively for the TE and TM modes. Here we have to consider the regular scattering solution which is defined as that solution which for $k \to 0$ turns into the free

solution. For $r \to \infty$, it describes a superposition of incoming and outgoing spherical waves and in our model it can be written in the form

$$\phi_l^{\rm sc}(k,r) = j_l(kr)\Theta(R-r) + \frac{1}{2}\left(f_l(k)h_l^{(2)}(kr) + f_l^*(k)h_l^{(1)}(kr)\right)\Theta(r-R), \tag{16}$$

where $j_l(x) = \sqrt{\pi/2x}J_{l+1/2}(x)$ and $h_l^{(1,2)}(x) = \sqrt{\pi/2x}H_{l+1/2}^{(1,2)}(x)$ are the spherical Bessel functions and $f_l(k)$ and $f_l^*(k)$ are the Jost function and its complex conjugate. For $r \neq R$ these are solutions of the radial wave equation. Imposing the matching conditions (2) and (5) on (16), the Jost functions can be determined separately for each polarization,

$$\begin{aligned} f_l^{\rm TE}(k) &= 1 - i\Omega k R^2 j_l(kR)h_l^{(1)}(kR), \\ f_l^{\rm TM}(k) &= 1 + i\frac{\Omega}{k} j_l'(kR){h_l^{(1)}}'(kR). \end{aligned} \tag{17}$$

The corresponding formulas for imaginary argument read

$$\begin{aligned} f_l^{\rm TE}(ik) &= 1 + \frac{\Omega}{k}\, s_l(kR)e_l(kR), \\ f_l^{\rm TM}(ik) &= k^2\left(1 - \frac{\Omega}{k}\, s_l'(kR)e_l'(kR)\right), \end{aligned} \tag{18}$$

where we used the modified Riccati-Bessel functions

$$s_l(x) = \sqrt{\frac{\pi x}{2}}\, I_{l+1/2}(x), \qquad e_l(x) = \sqrt{\frac{2x}{\pi}}\, K_{l+1/2}(x). \tag{19}$$

In (17) and (18) we made use of the freedom to multiply the Jost functions by a constant which does not influence the vacuum energy (15).

In zetafunctional regularization, the ultraviolet divergences manifest themselves as poles in the of the regularized energy $E_0(s)$, (15). In our case the pole structure reads

$$2\mu^{-2s}(4\pi)^{3/2}\Gamma\left(s-\frac{1}{2}\right)E_0(s) = \sum_{k\geq 0}\frac{a_{k/2}}{s-2+\frac{k}{2}} + \sum_{k\geq 3}\frac{a'_{k/2}}{\left(s-2+\frac{k}{2}\right)^2} + \dots . \tag{20}$$

and included the double poles which will appear below in the TM mode. Fortunately, the double poles start from $k=5$ and do not influence the renormalization.

In order to find the poles one has to construct the analytic continuation of $E_0(s)$ into the region where the sum and the integral in representation (15) do not converge. For this one may use the uniform asymptotic expansion $f_l^{\rm as}(ik)$ of the Jost function for large both, ν and k with $z \equiv \frac{k}{\nu}$ fixed. We define

$$E_0^{\rm as}(s) = -\frac{\cos\pi s}{\pi}\mu^{2s}\sum_{l=1}^{\infty}\nu\int_0^{\infty}dk\, k^{1-2s}\frac{\partial}{\partial k}\ln f_l^{\rm as}(ik), \tag{21}$$

whose pole contributions coincide with that of $E_0(s)$, (15). In the following subsections we obtain the heat kernel coefficients separately for the TE and TM modes. As for the TE modes the procedure is well known. One simply inserts the uniform asymptotic expansions of the Bessel functions entering (18) and the analytic continuation is an easy task. For the TM mode, however, this does not work and a more sophisticated treatment is in order.

A. The asymptotic expansion for the TE mode

Directly inserting the known uniform asymptotic expansions of the Bessel modified functions [1] into (18) one obtains with $k = \nu z$

$$f^{\rm TE}(ik) \simeq 1 + \frac{\Omega R t}{2\nu}\left(1 + \sum_{j\geq 1}\frac{c_j^{\rm TE}}{\nu^{2j}}\right). \tag{22}$$

The $c_j^{\rm TE}$ are polynomials in $t = 1/\sqrt{1+z^2}$ (see [3], Appendix A) and we used

$$c_1^{\rm TE} = \frac{t^2}{8}\left(1-6t^2+5t^4\right), \quad c_2^{\rm TE} = \frac{t^4(1-t^2)}{128}\left(27-553t^2+1617t^4-1155t^6\right). \tag{23}$$

Using this expansion we define the asymptotic part of the logarithm of the Jost function,

$$\ln f_l^{{\rm TE},as}(ik) = \sum_{i=1}^{3} \frac{D_i}{\nu^i}, \tag{24}$$

with

$$\begin{aligned} D_1 &= \frac{\Omega R t}{2}, \\ D_2 &= -\frac{(\Omega R)^2 t^2}{8}, \\ D_3 &= \frac{(\Omega R)^3 t^3}{24} + \frac{\Omega R t^3}{16}\left(1-5t^2\right)\left(1-t^2\right). \end{aligned} \tag{25}$$

Inserting this into (21) we define $E_0^{\rm TE,\,as}(s)$

$$E_0^{\rm TE,\,as}(s) = -\frac{\cos\pi s}{\pi}\mu^{2s}\sum_{l=1}^{\infty}\nu^{2-2s}\int\limits_0^{\infty} dz\, z^{1-2s}\frac{\partial}{\partial z}\ln f_l^{{\rm TE},as}(ik). \tag{26}$$

This expression is in a suitable form for the analytic continuation because it can be immediately expressed in terms of known functions. In (26), the sum over ν results in Hurwitz zeta functions,

$$\sum_{l=0}^{\infty}\nu^{-s} = \zeta_H\left(s;\frac{1}{2}\right) \tag{27}$$

for the scalar field with the s-wave included and, without the s-wave,

$$\sum_{l=1}^{\infty}\nu^{-s} = \zeta_H\left(s;\frac{3}{2}\right) \tag{28}$$

for the electromagnetic field. The integration over z can be carried out using

$$\int_0^{\infty} z^{1-2s}t^n dz = \frac{\Gamma(1-s)\Gamma(s-1+\frac{n}{2})}{2\Gamma(\frac{n}{2})} \qquad \left(1-\frac{n}{2}<s<1\right). \tag{29}$$

From Eq. (20) we get then the heat kernel coefficients for the TE mode. These are shown in Table I. We remind that for the TE polarization there are no double poles which can be confirmed by direct inspection in the above formulas. This holds also if higher orders of the asymptotic expansion from (22) are included. For the coefficients for the scalar field, i.e., including the s-wave, we confirm the results found in [7, 17]. The coefficients for the electromagnetic field become different starting from $k=2$.

k	$l=0,1,\dots$	$l=1,2,\dots$
0	0	0
1/2	0	0
1	$-4\pi\Omega R^2$	$-4\pi\Omega R^2$
3/2	$\pi^{3/2}\Omega^2R^2$	$\pi^{3/2}\Omega^2R^2$
2	$-\frac{2}{3}\pi\Omega^3R^2$	$-\frac{2}{3}\pi\Omega^3R^2+4\pi\Omega$

TABLE I: The first few heat kernel coefficients $a_k^{\rm TE}$ for the TE polarization, including the s-wave in the second column and without it in the third column.

B. The asymptotic expansion for the TM mode

As before in the preceding subsection we insert the asymptotic expansions of the Bessel functions into the Jost function (18) and obtain

$$f_l^{\rm TM}(ik) = z^2 + \frac{\Omega R}{2\nu t}\left(1+\sum_{j\geq 1}\frac{c_j^{\rm TM}}{\nu^{2j}}\right). \tag{30}$$

The coefficients $c_j^{\rm TM}$ are also polynomials in t and we need only

$$c_1^{\rm TM} = -\frac{t^2}{8}\left(1-6t^2+7t^4\right). \tag{31}$$

The direct insertion of this asymptotic expansion into $\ln f_l^{\rm TM}(ik)$ would produce powers of z in the denominator and a term-by-term integration over z would be impossible. Therefore we define

$$p = z^2 + \frac{\Omega R}{2\nu t} \tag{32}$$

and perform the asymptotic expansion of the logarithm for large ν with fixed p,

$$\ln f_l^{\rm TM}(ik) \sim \ln p + \ln\left(1+\frac{\Omega R}{2\nu t p}\sum_{i\geq 1}\frac{c_j^{\rm TM}}{\nu^{2j}}\right), \tag{33}$$

and define a part of this expansion as

$$\ln f_l^{\rm TM,\, as}(ik) = \ln p + \frac{\Omega R}{2tp}\frac{c_1^{\rm TM}}{\nu^3}. \tag{34}$$

In fact, this is a partial re-summation of the expansion one would obtain acting in the same way as in the preceding subsection. Obviously, the pole contributions to $E_0^{\rm TM}(s)$ can be obtained from this expansion too. The advantage of (34) is besides allowing for a term-by-term integration that it turns into the asymptotic expansion for Neumann boundary conditions in the formal limit $\Omega\to\infty$.

Inserting now (34) into (21) we obtain for the asymptotic part of the vacuum energy of the TM mode the expression

$$E_0^{\rm TM,\, as}(s) = \mu^{2s}\sum_{l\geq 1}\sum_{k=0}^{7}\sum_{n=1}^{2}\sum_{r=0}^{4} Y_{k,n}^r \nu^{2-2s-r} I_{k,n}^s\left(\frac{\Omega R}{2\nu}\right) \tag{35}$$

with

$$I_{k,n}^s(\alpha) = -\frac{\cos\pi s}{\pi}\int_0^\infty dz\,\frac{z^{2-2s}(1+z^2)^{-k/2}}{\left(z^2+\alpha\sqrt{1+z^2}\right)^n} \tag{36}$$

and the non-zero coefficients $Y_{k,n}^r$ are given in Eqs. (37):

$$\begin{array}{llllll}
Y_{0,1}^0 = 1, & Y_{1,1}^1 = \frac{Q}{4}, & & \\
Y_{3,1}^3 = \frac{Q}{32}, & Y_{5,1}^3 = -\frac{9Q}{16}, & Y_{7,1}^3 = \frac{35Q}{32}, \\
Y_{1,2}^3 = \frac{Q}{16}, & Y_{3,2}^3 = -\frac{3Q}{8}, & Y_{5,2}^3 = \frac{7Q}{16}, \\
Y_{2,2}^4 = \frac{Q^2}{64}, & Y_{4,1}^4 = -\frac{3Q^2}{32}, & Y_{6,2}^4 = \frac{7Q^2}{64},
\end{array} \tag{37}$$

with $Q = \Omega R$. As a result of the re-summations these integrals are more complicated than that which appeared for the TE modes.

In order to perform in (35) the analytic continuation in s we would like to expand the integrals (36) in a series in powers of α. The direct expansion in the integrand is impossible because of

the behavior for small z. Therefore the idea is to move the integration contour away from passing through $z=0$.

Splitting the cosine into two exponentials we change the integration in the second part by the substitution $z \to -z$. Both part can be united into one integral over the whole axis and (36) becomes

$$I^s_{k,n}(\alpha) = -\frac{e^{i\pi s}}{2\pi} \int\limits_{-\infty}^{\infty} dz\, \frac{z^{2-2s}(1+z^2)^{-k/2}}{\left(z^2+\alpha\sqrt{1+z^2}\right)^n} \,. \tag{38}$$

Next we move the integration path into the upper half plane. There is a pole of n-th order in iz_0. We move the integration path across this pole which gives an additional contribution which results in the second terms in (39), (40) and (41) below. Moving the path further upwards we hit the branch cut starting from $z=i$ which originates from $\sqrt{1+z^2}$ in (38). To handle the singular behavior in $z=i$ we divide the contour into two parts, one is a circle around $z=i$ with a small radius ϵ and the other is the path closed to the two branches of the cut with $z=ix$, $x=1+\epsilon\dots\infty$. Integrating in this integral by parts $k-2$ times, the surface terms just cancel the divergent terms coming from the circle around $z=i$ and in the limit $\epsilon\to 0$ we obtain an integral (first terms) and an explicit contribution (last terms) in Eqs. (39) and (40),

$$\begin{aligned} I^s_{2k+1,n}(\alpha) &= \frac{(-2)^{k+1}}{4\pi(2k-1)!!}\int_1^\infty dx\sqrt{x-1}F^{(k+1)}(x) \\ &+\frac{i^{2s-1}}{(n-1)!}\left[\frac{(z-iz_0)^n z^{2-2s}t^{2k+1}}{(z^2+\alpha\sqrt{z^2+1})^n}\right]^{(n-1)}_{z=iz_0} \\ &+\frac{i^{2s-1}}{(2k-1)!}\left[\frac{(i+z^2)^{2-2s}}{((i+z^2)^2+\alpha z\sqrt{2i+z})^n}\frac{1}{(2i+z^2)^{k+1/2}}\right]^{(2k-1)}_{z=0}, \end{aligned}\tag{39}$$

and

$$\begin{aligned} I^s_{2k,n}(\alpha) &= -\frac{(-2)^{k}}{4\pi(2k-3)!!}\int_1^\infty dx\sqrt{x-1}\Psi^{(k)}(x) \\ &+\frac{i^{2s-1}}{(n-1)!}\left[\frac{(z-iz_0)^n z^{2-2s}t^{2k}}{(z^2+\alpha\sqrt{z^2+1})^n}\right]^{(n-1)}_{z=iz_0} \\ &+\frac{i^{2s-1}}{(2k-2)!}\left[\frac{(i+z^2)^{2-2s}}{((i+z^2)^2+\alpha z\sqrt{2i+z})^n}\frac{1}{(2i+z^2)^{k}}\right]^{(2k-2)}_{z=0}, \end{aligned}\tag{40}$$

where we introduced the notations

$$\begin{aligned} z_0 &= \sqrt{-\frac{\alpha^2}{2}+\frac{\alpha}{2}\sqrt{\alpha^2+1}}, \\ F(x) &= x^{-s}\left[\frac{1}{(-x-i\alpha\sqrt{x-1})^n}+\frac{1}{(-x+i\alpha\sqrt{x-1})^n}\right], \\ \Psi(x) &= \frac{x^{-s}}{i\sqrt{x-1}}\left[\frac{1}{(-x-i\alpha\sqrt{x-1})^n}-\frac{1}{(-x+i\alpha\sqrt{x-1})^n}\right]. \end{aligned}$$

These formulas hold for $k\geq 0$ whereby for $k=0$ the last term in $I^s_{2k,n}(\alpha)$ must be dropped. As an example we note for the simplest case with $k=0$ and $n=1$,

$$I^s_{0,1}(\alpha) = \frac{\alpha}{2\pi}\int_1^\infty \frac{dx x^{-s+\frac{1}{2}}\sqrt{x-1}}{x^2+\alpha^2(x-1)} + \frac{2^{s-\frac{3}{2}}\alpha^{-s+\frac{1}{2}}}{\sqrt{\alpha^2+4}}\left(-\alpha+\sqrt{\alpha^2+4}\right)^{-s+\frac{3}{2}}. \tag{41}$$

The merit of the representations (39) and (40) is that these can be directly expanded into powers of α. For instance, from (41) we get

$$I^s_{0,1} \;=\; \alpha^{-s+\frac{1}{2}}\left[\frac{1}{2}+\frac{1}{4}(s-\frac{3}{2})\alpha+\frac{1}{16}(s-\frac{5}{2})(s-\frac{1}{2})\alpha^2+\frac{1}{96}(s-\frac{7}{2})(s-\frac{3}{2})(s+\frac{1}{2})\alpha^3+\dots\right]$$

$$+\frac{\alpha}{4\sqrt{\pi}}\left[\frac{\Gamma(s)}{\Gamma(s+\frac{3}{2})}-\frac{3\alpha^2}{2}\frac{\Gamma(s+1)}{\Gamma(s+\frac{7}{2})}+\dots\right]. \tag{42}$$

Now we insert these expansions into $E_0^{\mathrm{TM,\,as}}(s)$, (35) with $\alpha = \Omega R/(2\nu)$. There the sum over the orbital momentum $\nu = l + 1/2$ delivers directly Hurwitz zeta functions, $\zeta_H(a, 3/2)$, with corresponding a. In case the s-wave is included, the result would be expressed in terms $\zeta_H(a, 1/2)$. Keeping the necessary number of contributions we come to

$$E_0^{\mathrm{TM,\,as}}(s) = \frac{4(\mu R)^{2s}}{R}\left\{-\frac{Q(s-\frac{1}{2})\Gamma(s)}{8\sqrt{\pi}\Gamma(s+\frac{3}{2})}\zeta_H(2s-1,\frac{3}{2})\right. \tag{43}$$
$$+\frac{Q(s-\frac{1}{2})\left(3Q^2-(2s+3)(2s+5)((2s-1)(7s+\frac{31}{2})+27)\right)\Gamma(s+1)}{192\sqrt{\pi}\Gamma(s+\frac{7}{2})}\zeta_H(2s+1,\frac{3}{2})+\dots$$
$$+2^{s-\frac{3}{2}}Q^{-s+\frac{1}{2}}\left[\zeta_H(s-\frac{3}{2},\frac{3}{2})+\frac{Q(s-\frac{1}{2})}{4}\zeta_H(s-\frac{1}{2},\frac{3}{2})+\frac{\left((s-\frac{1}{2})Q^2+8\right)(s-\frac{1}{2})}{32}\zeta_H(s+\frac{1}{2},\frac{3}{2})\right.$$
$$\left.\left.+\frac{Q\left(\left((s-\frac{1}{2})^2-1\right)Q^2+24(s+\frac{17}{2})\right)s}{384}\zeta_H(s+\frac{3}{2},\frac{3}{2})+\dots\right]\right\}.$$

From this representation and together with Eq. (20) we obtain the heat kernel coefficients for the TM polarization. These are shown in Table II.

These coefficients can be compared with that obtained for a plane plasma sheet by dividing by the area of the sphere, $4\pi R^2$, and taking the limit $R\to\infty$. In fact, these coincide for $k \geq 1$ with that obtained in [9].

a_k^{TM}	$l = 0, 1, \dots$	$l = 1, 2, \dots$
0	0	0
1/2	$8\pi^{3/2}R^2$	$8\pi^{3/2}R^2$
1	$-\frac{4\pi}{3}\Omega R^2$	$-\frac{4\pi}{3}\Omega R^2$
3/2	$\frac{14}{3}\pi^{3/2}$	$-\frac{10}{3}\pi^{3/2}$
2	$-8\pi\Omega+\frac{2\pi}{15}\Omega^3R^2$	$-4\pi\Omega+\frac{2\pi}{15}\Omega^3R^2$

TABLE II: The heat kernel coefficients for TM polarization. We represent the calculation with s-wave (second column) and without s-wave (third column). The difference appears starting from $k = 3/2$.

From Eqs. (43) together with (20) also the double poles mentioned in the introduction follows. It comes at $s = -1/2$ from the zeta function $\zeta_H(s+\frac{3}{2};\frac{3}{2})$ in the last line in (35) and the gamma function in the left hand side of Eq. (20) and corresponds to heat kernel coefficient $a'_{5/2}$. There is no double pole at point $s = 1/2$ which corresponds to $a'_{3/2}$ due to factor $(s-1/2)$ at $\zeta_H(s+\frac{1}{2};\frac{3}{2})$ in (43). Therefore the logarithmic contributions for heat kernel expansion starts from $a'_{5/2}$.

IV. THE RENORMALIZED VACUUM ENERGY

The renormalization of the vacuum energy is given by Eqs. (13) and (10) with the heat kernel coefficients calculated in the preceding section. As discussed in Sec. II, it remains to dispose of the freedom of a finite renormalization. It turns out that it is possible to join this with the behavior for large Ω, i.e. with the limit of the plasma sphere to become an ideal conductor.

We start from dividing the regularized vacuum energy into two parts,

$$E_{\mathrm{vac}}(s) = E_{\mathrm{vac}}^{\mathrm{num}} + E_{\mathrm{vac}}^{\mathrm{as}}(s), \tag{44}$$

where $E_{\mathrm{vac}}^{\mathrm{as}}$ is defined by Eq.(21) and the 'numerical' parts of the energy are defined by

$$E_{\mathrm{vac}}^{\mathrm{num}} = -\frac{1}{\pi}\sum_{l=1}^{\infty}\nu\int_0^{\infty}dk\,k\frac{\partial}{\partial k}\left(\ln f_l(ik) - \ln f_l^{\mathrm{as}}(ik)\right). \tag{45}$$

In this way the asymptotic part of the logarithm of the Jost function, $\ln f_l^{\rm as}(ik)$, is subtracted in $E_{\rm vac}^{\rm num}$ and added back in $E_{\rm vac}^{\rm as}$. In general, the definition of $\ln f_l^{\rm as}(ik)$ is not unique. The only one has to ensure in any case is that the sum and the integral in (45) do converge if one puts $s=0$ there. We did this in Eq.(45). This part is called 'numerical' since it allows for a direct numerical evaluation.

In the preceding section we used the expressions (24) and (34) for $\ln f_l^{\rm as}(ik)$ for the calculation of the heat kernel coefficients. In this section we take the same expression for the TM mode (34) and different expression for TE mode, Eq.(46) below. In $\ln f_l^{\rm TE,\,as}(ik)$ we make a re-summation like that in $\ln f_l^{\rm TM,\,as}(ik)$, (34), in order to archive also for $\ln f_l^{\rm TE,\,as}(ik)$ the property in the formal limit $\Omega\to\infty$ to turn into the corresponding expression for an ideal conductor.

With this motivation we define

$$\ln f_l^{\rm TE,\,as}(ik)=\ln w+\frac{\Omega Rt}{2w}\frac{c_1^{\rm TE}}{\nu^3}, \tag{46}$$

where

$$w=1+\frac{\Omega Rt}{2\nu}. \tag{47}$$

We would like to stress again that a redefinition of $\ln f_l^{\rm as}(ik)$ is merely a redistribution of contributions between $E_{\rm vac}^{\rm num}$ and $E_{\rm vac}^{\rm as}(s)$. With the definitions (46) and (34) we archived that $E_{\rm vac}^{\rm num}$ in the limit $\Omega\to\infty$ must turn into the corresponding ideal conductor expressions. This follows because in (45) both, the Jost function and the part (46) of its asymptotic expansion, do that and because in addition the integral and the sum are convergent. Indeed, from the numerical evaluation we got

$$\begin{aligned}\lim_{Q\to\infty}E_{\rm vac}^{\rm TE,\,num} &= \frac{0.00090282}{R},\\ \lim_{Q\to\infty}E_{\rm vac}^{\rm TM,\,num} &= \frac{-0.00160178}{R},\end{aligned} \tag{48}$$

which is the same as if one takes conductor boundary conditions from the beginning. With the other part, $E_{\rm vac}^{\rm as}(s)$, this is not such simple. The corresponding calculations are carried out in the Appendix and the result is

$$\begin{aligned}E_{\rm vac}^{\rm TE,\,as}(s) &= -\frac{a_2^{\rm TE}}{32\pi^2}\left[\frac{1}{s}-2\ln\frac{\Omega}{2\mu}\right]+\frac{\Omega^3R^2}{72\pi}+\frac{\Omega}{180\pi}+E_{\rm vac}^{\rm TE,\,an}+O(s),\\ E_{\rm vac}^{\rm TM,\,as}(s) &= -\frac{a_2^{\rm TM}}{32\pi^2}\left[\frac{1}{s}-2\ln\frac{\Omega}{2\mu}\right]+\frac{7\Omega^3R^2}{1800\pi}-\frac{29\Omega}{36\pi}+E_{\rm vac}^{\rm TM,\,an}+O(s.\end{aligned} \tag{49}$$

These expressions are sums of a divergent part (it is proportional to the heat kernel coefficient a_2 as expected), two terms growing with Ω and an 'analytical' part,

$$\begin{aligned}E_{\rm vac}^{\rm TE,\,an} &= \sum_{l=1}^{4}\mathcal{V}_l+\sum_{l=1}^{3}\widetilde{\mathcal{V}}_l,\\ E_{\rm vac}^{\rm TM,\,an} &= \frac{1}{2}\mathcal{J}_1+\sum_{l=2}^{6}\mathcal{J}_l+\sum_{l=1}^{4}\widetilde{\mathcal{J}}_l.\end{aligned} \tag{50}$$

All quantities entering $E_{\rm vac}^{\rm TE,\,an}$ and $E_{\rm vac}^{\rm TM,\,an}$ are defined in the Appendix. The analytical parts have the 'necessary' limit for $\Omega\to\infty$,

$$\begin{aligned}\lim_{Q\to\infty}E_{\rm vac}^{\rm TE,\,an} &= \frac{17}{128R},\\ \lim_{Q\to\infty}E_{\rm vac}^{\rm TM,\,an} &= -\frac{11}{128R}.\end{aligned} \tag{51}$$

With Eqs. (49) and the property (51) we have all information we need to complete the renormalization. We remind the discussion in Sec. II that all terms which are proportional to R, R^2

or which do not depend on R can be removed by a redefinition of the parameters in the classical part. In (49) this concerns all except the last ones, $E_{\rm vac}^{\rm TE,\,an}$ and $E_{\rm vac}^{\rm TM,\,an}$. That means, that we not only can remove the contribution proportional to a_2, but also the terms growing with Ω. For this reasons we define the renormalized vacuum energies by

$$\begin{aligned} E_{\rm vac}^{\rm TE,\,ren} &= E_{\rm vac}^{\rm TE,\,num} + E_{\rm vac}^{\rm TE,\,an}, \\ E_{\rm vac}^{\rm TM,\,ren} &= E_{\rm vac}^{\rm TM,\,num} + E_{\rm vac}^{\rm TM,\,an}. \end{aligned} \tag{52}$$

With these formulas we completed the model consisting of the classical energy and the vacuum energy which is the sum of the two contributions in (52). The main merit of this vacuum energy is that it turns for $\Omega \to \infty$ into the ideal conductor limit. Using the formulas for the Jost functions and their asymptotic parts and also the formulas in the Appendix, it is possible to evaluate this vacuum energy numerically. The results are shown in the figures IV and IV for the dimensionless function $\mathcal{E}$ defined by

$$E_{\rm vac}^{\rm ren} = \frac{\mathcal{E}(\Omega R)}{R} \tag{53}$$

as functions of their arguments $x = \Omega R$.

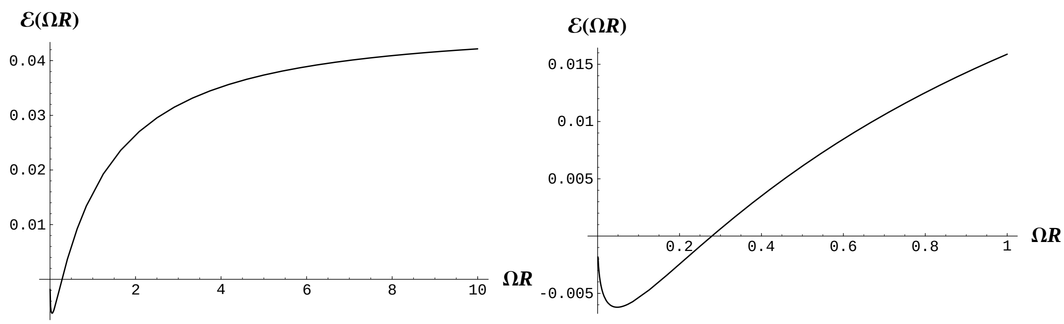

FIG. 1: The function $\mathcal{E}(\Omega R) = R E_{\rm vac}^{\rm ren}$ plotted as function of Ω. For large Ω it tends to the ideal conductor limit, $\lim_{\Omega R\to\infty} = 0.0046$ (left panel). For small Ω (right panel) it takes negative values and decreases as $\mathcal{E}(\Omega R) \sim -0.0589\sqrt{\Omega R}$.

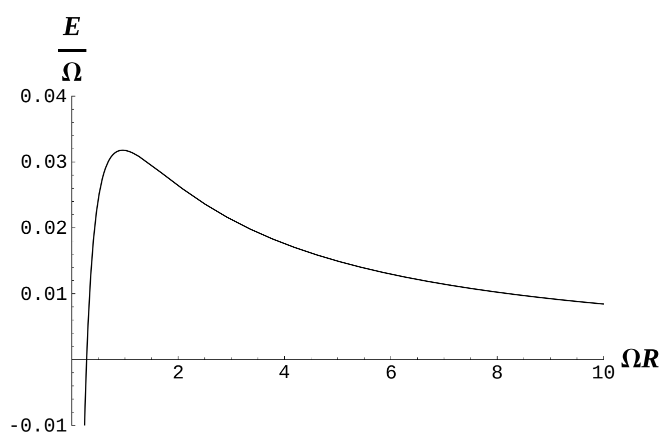

FIG. 2: The renormalized vacuum energy divided by Ω, $E_{\rm vac}^{\rm ren}/\Omega$, as a function of R. For large R it approaches the ideal conductor limit and for small radii, $R \lesssim \Omega^{-1}$, it becomes attractive.

It is also interesting to consider the limit of small argument of these functions which is equivalent to a small plasma frequency Ω. For the TE mode the main contributions come from the integrals

$$\mathcal{V}_1(\Omega R) = \frac{\Omega R \ln \Omega R}{48\pi} + O(\Omega R),$$

$$\begin{aligned}
\mathcal{V}_3(\Omega R) &= \frac{\Omega R \ln \Omega R}{48\pi} + O(\Omega R), \\
\widetilde{\mathcal{V}}_1(\Omega R) &= -\frac{\Omega R \ln \Omega R}{4\pi} + O(\Omega R).
\end{aligned}$$

For the TM mode we get accordingly

$$\begin{aligned}
\mathcal{J}_2(\Omega R) &= \frac{3(4-\sqrt{2})\zeta_R(\frac{5}{2})}{64\pi^2\sqrt{2}}\sqrt{\Omega R} + O(\Omega R \ln \Omega R), \\
\mathcal{J}_5(\Omega R) &= \sqrt{\Omega R}\frac{(1-\sqrt{2})\zeta_R(\frac{1}{2})}{8\sqrt{2}} + O(\Omega R \ln \Omega R), \\
\widetilde{\mathcal{J}}_1(\Omega R) &= -\frac{\sqrt{\Omega R}}{4} + O(\Omega R \ln \Omega R), \\
\widetilde{\mathcal{J}}_3(\Omega R) &= \frac{\sqrt{\Omega R}}{8} + O(\Omega R \ln \Omega R),
\end{aligned}$$

where $\zeta_R(x)$ is the Riemann zeta function. The numerical parts are $\sim O(\Omega)$ which is easy to show using the next term of asymptotic expansion of the Jost functions. In this way we obtain the following behavior of the vacuum energy for small Ω,

$$\begin{aligned}
E^{\mathrm{TE,\,ren}}|_{\Omega\to 0} &\simeq -\frac{5}{24\pi}\Omega \ \ln \Omega R, \\
E^{\mathrm{TM,\,ren}}|_{\Omega\to 0} &\simeq \left(\frac{3(4-\sqrt{2})\zeta(\frac{5}{2})}{64\pi^2\sqrt{2}} + \frac{(1-\sqrt{2})\zeta(\frac{1}{2})}{8\sqrt{2}} - \frac{1}{8}\right)\sqrt{\frac{\Omega}{R}} \\
&= -0.0598\sqrt{\frac{\Omega}{R}}.
\end{aligned} \tag{54}$$

Therefore the main contribution for the energy comes from the TM polarization. The energy tends to zero proportional to $\sqrt{\Omega}$. The same behavior was observed in Refs. [9] and [5].

V. CONCLUSION

In the foregoing sections we considered the vacuum energy of the electromagnetic field interacting with a spherical plasma shell. We calculated the heat kernel coefficients for both polarizations. For the TE case the standard methods apply, for the TM case a re-expansion of the asymptotic expansion of the logarithm of the Jost function was helpful. It turned out that the vacuum energy in zeta functional regularization and, with it the corresponding zeta function, have double poles. This implies that the corresponding spectral problem is not elliptic. On the other hand, at least based on the calculations carried out in this paper, there is nothing which would diminish the reasonability of this model.

A basic concern of this paper is to construct a model allowing for a physically meaningful interpretation of the renormalization. We considered with the breathing mode of the shell the simplest model for the classical motion of the shell. It turned out that this model is able to accommodate all renormalizations which we were like to carry out. These are the removal of the pole in s, i.e., of the ultraviolet divergence, and the removal of all contributions growing together with the plasma frequency Ω. It should be mentioned that this includes also the removal of the arbitrary constant μ which came in with the regularization. The nontrivial statement which allowed for doing so is that the dependence on the radius R of all these contributions is polynomial not exceeding R^2.

It would be interesting to investigate the question whether this procedure can be carried out also for more general deformations of the shell. In principle, most ingredients for such a calculation are available. Especially, the heat kernel coefficients for the TE modes can be taken from [17]. It would remain to calculate the coefficients for the TM modes.

Concerning the arbitrariness of the normalization procedure we would like to mention that the removal of the contributions growing together with Ω can be considered as a normalization condition. It ensures the uniqueness of the renormalized vacuum energy and makes this model

physically meaningful. In this way, the gap between the renormalization procedure in quantum field theory in smooth background fields and the removal of divergences of the Casimir energy in the background of boundaries, as suggested for example in [4] (section 6.5), is narrowed. At once in this way the much discussed vacuum energy of a conducting spherical shell now appears as a limiting case of a slightly more physical model.

Acknowledgments

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VI. APPENDIX

In this appendix we perform the analytic continuation in the asymptotic parts of the vacuum energy defined in section IV. For that, we use the following integral representations of some sums,

$$\begin{aligned}
\sum_{l=0} \frac{\nu^{3-2s}}{\nu+a} &= -\frac{\pi a^{3-2s}}{\sin 2\pi s} + 2\int_0^\infty \frac{dy y^{3-2s}}{y^2+a^2}\frac{-a\cos\pi s + y\sin\pi s}{1+e^{2\pi y}}, \ \frac{3}{2} < \Re s < 2, \qquad (55)\\
\sum_{l=0} \frac{\nu^{2-2s}}{\nu+a} &= \frac{\pi a^{2-2s}}{\sin 2\pi s} + 2\int_0^\infty \frac{dy y^{2-2s}}{y^2+a^2}\frac{a\sin\pi s + y\cos\pi s}{1+e^{2\pi y}}, \ 1 < \Re s < \frac{3}{2},\\
\sum_{l=0} \frac{\nu^{1-2s}}{\nu+a} &= -\frac{\pi a^{1-2s}}{\sin 2\pi s} - 2\int_0^\infty \frac{dy y^{1-2s}}{y^2+a^2}\frac{-a\cos\pi s + y\sin\pi s}{1+e^{2\pi y}}, \ \frac{1}{2} < \Re s < 1,\\
\sum_{l=0} \frac{\nu^{1-2s}}{(\nu+a)^2} &= -\frac{2\pi(s-\frac{1}{2})a^{-2s}}{\sin 2\pi s} - 2\int_0^\infty \frac{dy y^{-2s}}{(y^2+a^2)^2}\frac{2ay\sin\pi s - (a^2-y^2)\cos\pi s}{1+e^{2\pi y}}, \ \Re s < 1,\\
\sum_{l=0} \frac{1}{(\nu+a)^2} &= \pi a \int_0^\infty \frac{dy}{y^2+a^2}\frac{1}{\cosh^2\pi y},
\end{aligned}$$

which were obtained using the Abel-Plana formula in the form

$$\sum_{l=0}^\infty f(l+\frac{1}{2}) = \int_0^\infty dx\ f(x) - i\int_0^\infty dx\ \frac{f(iy)-f(-iy)}{1+e^{2\pi y}}.$$

A. TE polarization. $b = Qt/2$

The expression for asymptotic part $E_{\rm vac}^{\rm TE,\,as}$ appearing from inserting (46) into (26), after carrying out the differentiation, has the following form

$$\begin{aligned}
E_{\rm vac}^{\rm TE,\,as} &= \frac{\cos\pi s}{\pi R}(\mu R)^{2s}\sum_{l=1}^\infty \nu^{2-2s}\int_0^\infty dz z^{2-2s}\\
&\times \left\{\frac{Qt^3}{2w\nu} + \frac{Qt^5\left(3-30t^2+35t^4\right)}{16w\nu^3} - \frac{Q^2t^6\left(1-6t^2+5t^4\right)}{32w^2\nu^4}\right\},
\end{aligned}$$

with $w = 1+Qt/2\nu$ and $Q = \Omega R$. We perform the calculations separately for the contribution from each power of ν using the formulas (55) given above.

$$[\nu^{-1}] \ : \ \frac{\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{Qt^3}{2w\nu}$$

$$
\begin{aligned}
&= Q\frac{\cos\pi s}{2\pi}(\mu R)^{2s}\int_0^\infty dz z^{2-2s}t^3\sum_{l=1}\frac{\nu^{2-2s}}{\nu+b} \\
&= \frac{1}{48\pi}\left(Q^3-\frac{11}{2}Q\right)\left[\frac{1}{s}-2\ln\frac{\Omega}{2\mu}\right]+\frac{Q^3}{72\pi}+\mathcal{V}_1(Q)+\mathcal{V}_2(Q)+\widetilde{\mathcal{V}}_1(Q), \\
[\nu^{-3}] \ \ : \ \ & \frac{\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{Qt^5\left(3-30t^2+35t^4\right)}{16w\nu^3} \\
&= Q\frac{\cos\pi s}{16\pi}(\mu R)^{2s}\int_0^\infty dz z^{2-2s}t^5\left(3-30t^2+35t^4\right)\sum_{l=1}\frac{\nu^{-2s}}{\nu+b} \\
&= -\frac{Q}{96\pi}\left[\frac{1}{s}-2\ln\frac{\Omega}{2\mu}\right]+\frac{Q}{5040\pi}+\mathcal{V}_3(Q)+\widetilde{\mathcal{V}}_2(Q), \\
[\nu^{-4}] \ \ : \ \ & -\frac{\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{Q^2t^6\left(1-6t^2+5t^4\right)\epsilon^4}{32w^2} \\
&= -\frac{Q^2}{32\pi}\int_0^\infty dz z^2t^6\left(1-6t^2+5t^4\right)\sum_{l=1}\frac{1}{(\nu+b)^2} \\
&= \frac{3Q}{560\pi}+\mathcal{V}_4(Q)+\widetilde{\mathcal{V}}_3(Q),
\end{aligned}
$$

where the following integrals were introduced,

$$
\begin{aligned}
\mathcal{V}_1 &= -\frac{Q}{2\pi}\int_0^\infty\frac{ydy}{1+e^{2\pi y}}\ln\left[1+\frac{4y^2}{Q^2}\right], \\
\mathcal{V}_2 &= -\frac{Q}{\pi}\int_0^\infty\frac{y^3dy}{1+e^{2\pi y}}\int_0^\infty\frac{xt^4dx}{y^2+Q^2t^2/4}\frac{1}{1+xt}, \\
\mathcal{V}_3 &= \frac{Q}{8\pi}\int_0^\infty\frac{ydy}{1+e^{2\pi y}}\int_0^\infty\frac{x^2t^5(3-30t^2+35t^4)dx}{y^2+Q^2t^2/4}, \\
\mathcal{V}_4 &= \frac{Q}{16}\int_0^\infty\frac{y^2dy}{\cosh^2\pi y}\int_0^\infty\frac{x^2t^5(1-6t^2+5t^4)dx}{y^2+Q^2t^2/4} \\
\widetilde{\mathcal{V}}_1(Q) &= \frac{Q^2}{4\pi}\int_0^\infty\frac{x^2t^4dx}{1+Qt}+\frac{Q}{4\pi}(1-\ln 2Q) \\
&= \frac{1}{8\pi}\left(\pi-4Q+i\pi\sqrt{-1+4Q^2}+2\sqrt{-1+4Q^2}\operatorname{arctanh}\left[\frac{2Q}{\sqrt{-1+4Q^2}}\right]\right) \\
&+ \frac{Q}{4\pi}(1-\ln 2Q), \\
\widetilde{\mathcal{V}}_2(Q) &= -\frac{Q}{8\pi}\int_0^\infty dx x^2t^5\frac{3-30t^2+35t^4}{1+Qt}, \\
\widetilde{\mathcal{V}}_3(Q) &= \frac{Q^2}{8\pi}\int_0^\infty dx x^2t^6\frac{1-6t^2+5t^4}{(1+Qt)^2}.
\end{aligned}
$$

We note the following expressions which are necessary to consider the ideal conductor limit,

$$
\begin{aligned}
\lim_{Q\to\infty}\widetilde{\mathcal{V}}_1(Q) &= \frac{1}{8}, \\
\lim_{Q\to\infty}\widetilde{\mathcal{V}}_2(Q) &= \frac{1}{256}, \\
\lim_{Q\to\infty}\widetilde{\mathcal{V}}_3(Q) &= \frac{1}{256}.
\end{aligned}
$$

All other integrals vanish in this limit.

B. TM polarization. $a = Q/2tz^2$

The expression for asymptotic part $E_{\rm vac}^{\rm TM,\,as}$ appearing from inserting (34) into (26), after carrying out the differentiation, has the following form

$$
\begin{aligned}
E_{\rm vac}^{\rm TM,\,as} &= -\frac{2\cos\pi s}{\pi R}(\mu R)^{2s}\sum_{l=1}^{\infty}\nu^{2-2s}\int_0^\infty dz z^{2-2s} \\
&\times \left\{\frac{1}{p}+\frac{Qt}{4p\nu}+\frac{Qt}{32\nu^3}\left(\frac{\left(35t^4-18t^2+1\right)t^2}{p}+\frac{14t^4-12t^2+2}{p^2}\right)+\frac{Q^2\left(7t^4-6t^2+1\right)t^2}{64p^2\nu^4}\right\}
\end{aligned}
$$

$p = z^2 + Q/2\nu t$. As before, we perform the calculations separately for the contribution from each power of ν using the formulas (55) given above:

$$
\begin{aligned}
[\nu^0] \quad &: \quad -\frac{2\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{1}{p} \\
&= -\frac{2\cos\pi s}{\pi}(\mu R)^{2s}\int_0^\infty dz z^{-2s}\sum_{l=1}\frac{\nu^{3-2s}}{\nu+a} \\
&= \left(-\frac{Q^3}{40\pi}-\frac{11Q}{48\pi}\right)\left[\frac{1}{s}-2\ln\frac{\Omega}{2\mu}\right]-\frac{Q^3}{100\pi}+\mathcal{J}_1(Q)+\mathcal{J}_2(Q)+\widetilde{\mathcal{J}}_1(Q), \\
[\nu^{-1}] \quad &: \quad -\frac{2\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{Qt}{4\nu p} \\
&= -Q\frac{\cos\pi s}{2\pi}(\mu R)^{2s}\int_0^\infty dz z^{-2s}t\sum_{l=1}\frac{\nu^{2-2s}}{\nu+a} \\
&= \left(\frac{Q^3}{48\pi}+\frac{11Q}{96\pi}\right)\left[\frac{1}{s}-2\ln\frac{\Omega}{2\mu}\right]+\frac{Q^3}{72\pi}-\frac{1}{2}\mathcal{J}_1(Q)+\mathcal{J}_3(Q)+\widetilde{\mathcal{J}}_2(Q), \\
[\nu^{-3}] \quad &: \quad -\frac{2\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{Qt}{32\nu^3}\left[\frac{t^2(35t^4-18t^2+1)}{p}+\frac{2(7t^4-6t^2+1)}{p^2}\right] \\
&= -Q\frac{\cos\pi s}{16\pi}(\mu R)^{2s}\int_0^\infty dz z^{2-2s}t\left[\frac{t^2(35t^4-18t^2+1)}{z^2}\sum_{l=1}\frac{\nu^{-2s}}{\nu+a}+\frac{2(7t^4-6t^2+1)}{z^4}\sum_{l=1}\frac{\nu^{1-2s}}{(\nu+a)^2}\right] \\
&= \frac{23Q}{96\pi}\left[\frac{1}{s}-\ln\frac{\Omega}{2\mu}\right]-\frac{547Q}{720\pi}+\mathcal{J}_4(Q)+\mathcal{J}_5(Q)+\widetilde{\mathcal{J}}_3(Q), \\
[\nu^{-4}] \quad &: \quad -\frac{2\cos\pi s}{\pi}(\mu R)^{2s}\sum_{l=1}\nu^{2-2s}\int_0^\infty dz z^{2-2s}\frac{Q^2t^2(7t^4-6t^2+1)}{64\nu^4p^2} \\
&= -\frac{Q^2}{32\pi}\int_0^\infty dz z^{-2}t^2(7t^4-6t^2+1)\sum_{l=1}\frac{1}{(\nu+a)^2} \\
&= -\frac{11Q}{240\pi}+\mathcal{J}_6(Q)+\widetilde{\mathcal{J}}_4(Q),
\end{aligned}
$$

where the following integrals were introduced,

$$
\begin{aligned}
\mathcal{J}_1(Q) &= -\frac{Q^3}{16\pi}\int_0^\infty\frac{zdz}{1+e^{\pi Qz/2}}\left[\frac{\operatorname{arctanh}\sqrt{1-z^2}}{\sqrt{1-z^2}}+\ln\frac{z}{2}\right], \\
\mathcal{J}_2(Q) &= \frac{2Q}{\pi}\int_0^\infty\frac{y^3dy}{1+e^{2\pi y}}\int_0^\infty\frac{x^2tdx}{x^4y^2+Q^2/4t^2}\frac{1}{1+xt}, \\
\mathcal{J}_3(Q) &= \frac{Q}{\pi}\int_0^\infty\frac{y^3dy}{1+e^{2\pi y}}\int_0^\infty\frac{x^3t^2dx}{x^4y^2+Q^2/4t^2}\frac{1}{1+xt}, \\
\mathcal{J}_4(Q) &= \frac{Q}{8\pi}\int_0^\infty dxt^3x^4(35t^4-18t^2+1)\int_0^\infty\frac{1}{y^2x^4+Q^2/4t^2}\frac{ydy}{1+e^{2\pi y}},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_5(Q) &= \frac{Q}{4\pi}\int_0^\infty dxtx^2(7t^4-6t^2+1)\int_0^\infty \frac{x^4y^2-Q^2/4t^2}{(y^2x^4+Q^2/4t^2)^2}\frac{ydy}{1+e^{2\pi y}},\\
\mathcal{J}_6(Q) &= \frac{Q}{16}\int_0^\infty dxt^3x^4(7t^4-6t^2+1)\int_0^\infty \frac{dy}{y^2x^4+Q^2/4t^2}\frac{y^2}{\cosh^2\pi y},\\
\widetilde{\mathcal{J}}_1(Q) &= -\frac{Q}{2\pi}\int_0^\infty \frac{xtdx}{(x^2+Q/t)(x+Q)}\frac{1}{1+xt},\\
\widetilde{\mathcal{J}}_2(Q) &= -\frac{Q^2}{4\pi}\int_0^\infty \frac{dx}{x^2+Q/t}+\frac{Q}{4\pi}\ln 2Q,\\
\widetilde{\mathcal{J}}_3(Q) &= \frac{Q}{8\pi}\int_0^\infty dxx^2\left[\frac{t^3(35t^4-18t^2+1)}{x^2+Q/t}+\frac{2t(7t^4-6t^2+1)}{(x^2+Q/t)^2}\right],\\
\widetilde{\mathcal{J}}_4(Q) &= \frac{Q^2}{8\pi}\int_0^\infty \frac{dxt^2x^2(7t^4-6t^2+1)}{(x^2+Q/t)^2}.
\end{aligned}
$$

We note the following expressions which are necessary to consider the ideal conductor limit,

$$
\begin{aligned}
\lim_{Q\to\infty}\widetilde{\mathcal{J}}_2(Q) &= -\frac{1}{8},\\
\lim_{Q\to\infty}\widetilde{\mathcal{J}}_3(Q) &= \frac{7}{256},\\
\lim_{Q\to\infty}\widetilde{\mathcal{J}}_4(Q) &= \frac{3}{256},
\end{aligned}
$$

All other integrals vanish in this limit.

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